MID-SEMESTER EXAMINATION
M. MATH II YEAR, II SEMESTER September 29, 2010
Operator Theory

1. Obtain the spectral decomposition of the following matrices, that is, write them as unitary conjugates of diagonal matrices and also, write them as linear combinations of projections.

$$M = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}, N = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}$$

Solution: We consider the matrix M only. Observe that it is self-adjoint. First calculate the eigenvalues by

$$det(M - \lambda I) = 0.$$

Solving, the above equation we get two different eigenvalues $\lambda = 3$ and $\lambda = 7$. We now find the eigenspaces correspoding to these eigenvalues. For $\lambda = 3$, we have:

$$(M - 3I)u = 0,$$

where $u = (x, y)^t$, that is the colum. Then we have that the eigenspace corresponding to 3 is the one dimensional space spanned by the unit vector

$$e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Similarly, the eigenspace of $\lambda = 7$ is spanned by

$$e_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, M is expressed in the diagonal form as

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Further

$$M = 3P_1 + 7P_2,$$

where $P - 1$ is the proejction matrix $e_1 e_1^t$ and $P - 2$ is $e_2 e_2^t$ (product of the vector and its transpose) which can be comupted.

2. Let X and Y be Banach spaces. Show that the space $\mathcal{K}(X,Y)$ of all compact operators from X to Y is closed subspace of $L(X,Y)$, the space of all bounded operaotrs from X to Y.

Solution: This is a standard theorem. See any book on first course in functional analysis (Bhatia, Rudin or Kreyzig.)

3. Let $\mathcal{F}$ be the algebra of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

where a, b are complex numbers. Show that $\mathcal{F}$ with norm

$$\| \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \| = |a| + |b|$$

is a commutative Banach algebra. Compute the spectrum of an element in this algebra.

Solution: Banach algebra axioms are easy to verify. Clearly the identy matrix is the identity element here. It is also a Banach space since if

$$\begin{pmatrix} a_n & b_n \\ 0 & a_n \end{pmatrix}$$

is a Cauchy sequence, then by the above norm defintion, the sequences a_n and b_n shoud converge to some c and d respectively.. So the Cauchy sequence converges to

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

in the norm of $\mathcal{F}$. It is easy also to see that $\|AB\| \leq \|A\|\|B\|$ by straightforward calculation.

Now, we compute the spectrum of a member

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

If I is the identity operator, then spectrum of A is given by all λ such that

$$\begin{pmatrix} a-\lambda & b \\ 0 & a-\lambda \end{pmatrix}$$

has the 0 determinant. Hence the spectrum of A consists of the single scalar a.

4. Let $\mathcal{A}$ be a unital Banach algebra. Let $a, b \in \mathcal{A}$.
(i) Show that $1 - ab$ is invertible if and only if $1 - ba$ is so.
(ii) Show the spetrum staifies $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.

Solution: Let $c = (1 - ba)^{-1}$. Then $a(1 - ba)cb = ab$. This gives $1 - ab + acb - abacb = 1 - ab$. Pulling $-ab$ to the left, we get $(1 - ab)(1 + acb) = 1$. Similarly $(1 + acb)(1 - ab) = 1$. Hence (i) follows.

For (ii), proceed as follows which is simple. From the above, it follows that, for $\lambda \neq 0$, $\lambda - ab$ is invertible if and only if $\lambda - ba$ is invertible. Hence non-zero elements of $\sigma(ab)$ and that of $\sigma(ba)$ are the same. Including, the zero in both, we get (ii).

5. Let $\mathcal{E} = C[0, 1]$ the Banach algebra of all continuous complex functions on the interival. Show that for the ideal

$$I = \{f \in \mathcal{E} : \ f(0) = f(1) = 0\},$$

the quotient space $\mathcal{E}/I$ is isomorphic to $\mathbb{C}^2$.

Solution: This is a standard result, see a book on functional analysis. The proof is similar to proving that the kernel of a continuous linear functional is a maximal subspace, i.e. it has co-dimension 1.

6. Let $\mathcal{A}$ be a unital commutaive Banach algebra. Define the Gelfand map for $\mathcal{A}$, and show that it is a contarctive homomorphism.

Solution: Standard result.

7. Consider the set of question 4. Let $\lambda.1 = ab - ba$ for some $a, b \in \mathcal{A}$ and scalar λ. Show that λ is zero.

Solution: We understood that non-zero elements of $\sigma(ab)$ and $\sigma(ba)$ are the same. Now $\lambda.1 + ba = ab$ gives $\sigma(ab) = \lambda + \sigma(ba)$ by the spectral mapping theorem. This forces λ to be zero.